

Beta-representations of 0 and Pisot numbers

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Abstract

Let $\beta > 1$, d a positive integer, and

$$Z_{\beta,d} = \{z_1 z_2 \cdots \mid \sum_{i \geq 1} z_i \beta^{-i} = 0, z_i \in \{-d, \dots, d\}\}$$

be the set of infinite words having value 0 in base β on the alphabet $\{-d, \dots, d\}$. Based on a recent result of Feng on spectra of numbers, we prove that if the set $Z_{\beta, [\beta]-1}$ is recognizable by a finite Büchi automaton then β is a Pisot number. As a consequence of previous results, the set $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for every positive integer d if and only if $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for one $d \geq [\beta] - 1$. These conditions are equivalent to the fact that β is a Pisot number. The bound $[\beta] - 1$ cannot be further reduced.

1 Introduction

In the following β is a real number > 1 . The so-called *beta-numeration* has been introduced by Rényi in [8], and since then there are been many works in this domain, in connection with number theory, dynamical systems, and automata theory, see the survey [7] or more recent [9] for instance.

By a greedy algorithm each number of the interval $[0, 1]$ is given a β -expansion, which is an infinite word on a canonical alphabet of non-negative digits. When β is an integer, we obtain the classical numeration systems. When β is not an integer, a number may have different β -representations. The β -expansion obtained by the greedy algorithm is the greatest in the lexicographic ordering. The question of converting a β -representation into another one is equivalent to the study of the β -representations of 0. We

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focus on the question of the recognizability by a finite automaton of the set of β -representations of 0.

From this automaton it is possible to derive transducers, called digit-conversion transducers, that relate words with same values but written differently on the same or distinct alphabets of digits. The normalisation is a particular digit-conversion such that the result is the greedy expansion of the number considered.

Let d be a positive integer, and let

$$Z_{\beta,d} = \{z_1 z_2 \cdots \mid \sum_{i \geq 1} z_i \beta^{-i} = 0, z_i \in \{-d, \dots, d\}\}$$

be the set of infinite words having value 0 in base β on the alphabet $\{-d, \dots, d\}$.

The following result has been formulated in [7]:

Theorem 1 *The following conditions are equivalent:*

1. *the set $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for every integer d ,*
2. *the set $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for one integer $d \geq \lceil \beta \rceil$,*
3. *β is a Pisot number.*

(3) implies (1) is proved in [5], (1) implies (3) is proved in [1] and (2) implies (1) is proved in [6].

Recently Feng answered an open question raised by Erdős on accumulation points of the set

$$Y_d(\beta) = \left\{ \sum_{i=0}^n z_i \beta^i \mid n \in \mathbb{N}, z_i \in \{-d, \dots, d\} \right\}.$$

Theorem 2 ([4]) *Let $\beta > 1$. Then $Y_d(\beta)$ is dense in \mathbb{R} if and only if $\beta < d + 1$ and β is not a Pisot number.*

In the present note we use the Feng's result to simplify the proof of the implication (2) \Rightarrow (3) of Theorem 1 and moreover to replace the inequality $d \geq \lceil \beta \rceil$ by $d \geq \lceil \beta \rceil - 1$. In particular, we prove the conjecture stated in [7]:

If the set $Z_{\beta, \lceil \beta \rceil - 1}$ is recognizable by a finite Büchi automaton then β is a Pisot number.

Note that the value $d = \lceil \beta \rceil - 1$ is the best possible as $Z_{\beta,d}$ is empty if $d < \lceil \beta \rceil - 1$.

2 Preliminaries

2.1 Words and automata

Let A be a finite alphabet. A *finite word* w on A is a finite concatenation of letters from A , $w = w_1 \cdots w_n$ with w_i in A . The set of all finite words over A is denoted by A^* . An *infinite word* w on A is an infinite concatenation of letters from A , $w = w_1 w_2 \cdots$ with w_i in A . The set of all infinite words over A is denoted by $A^\mathbb{N}$. The infinite concatenation $uuu \cdots$ is noted u^ω . If $w = uv$, u is a *prefix* of w .

An *automaton* $\mathcal{A} = (A, Q, I, T)$ over A is a directed graph labeled by letters of the alphabet A , with a denumerable set Q of vertices called *states*. $I \subseteq Q$ is the set of *initial* states, and $T \subseteq Q$ is the set of *terminal* states. The automaton is said to be *finite* if the set of states Q is finite.

A finite path of \mathcal{A} is *successful* if it starts in I and terminates in T . The set of finite words *recognized* by \mathcal{A} is the set of labels of its successful finite paths.

An infinite path of \mathcal{A} is *successful* if it starts in I and goes infinitely often through T . The set of infinite words *recognized* by \mathcal{A} is the set of labels of its successful infinite paths. An automaton used to recognize infinite words in this sense is called a *Büchi automaton*.

2.2 Beta-numeration

We now recall some definitions and results on the so-called *beta-numeration*, see [7] or [9] for a survey. Let $\beta > 1$ be a real number. Any real number $x \in [0, 1]$ can be represented by a greedy algorithm as $x = \sum_{i=1}^{+\infty} x_i \beta^{-i}$ with x_i in the *canonical alphabet* $A_\beta = \{0, \dots, \lceil \beta \rceil - 1\}$ for all $i \geq 1$. The greedy sequence $(x_i)_{i \geq 1}$ corresponding to a given real number x is the greatest in the lexicographical order, and is said to be the β -*expansion* of x , see [8]. It is denoted by $d_\beta(x) = (x_i)_{i \geq 1}$. When the expansion ends in infinitely many 0's, it is said *finite*, and the 0's are omitted. The β -expansion of 1 is denoted $d_\beta(1) = (t_i)_{i \geq 1}$.

A β -*representation* of 0 on an alphabet $\{-d, \dots, d\}$ is an infinite sequence $z_1 z_2 \cdots$ of letters from this alphabet such that $\sum_{i \geq 1} z_i \beta^{-i} = 0$. If $d < \lceil \beta \rceil - 1$, then 0 has in the alphabet $\{-d, \dots, d\}$ only the trivial β -representation. On the other hand, $(-1)t_1 t_2 \cdots$ is a nontrivial β -representation of 0 on the alphabet $\{-\lceil \beta \rceil + 1, \dots, \lceil \beta \rceil - 1\}$, and thus we consider only alphabets $\{-d, \dots, d\}$ with $d \geq \lceil \beta \rceil - 1$.

Note that, if $Z_{\beta,d}$ is recognizable by a finite Büchi automaton, then, for every $c < d$, $Z_{\beta,c} = Z_{\beta,d} \cap \{-c, \dots, c\}^\mathbb{N}$ is recognizable by a finite Büchi

automaton as well.

Example 1 Take $\beta = \varphi = \frac{1+\sqrt{5}}{2}$ the Golden Ratio. It is a Pisot number, with $d_\varphi(1) = 11$. A finite Büchi automaton recognizing $Z_{\varphi,1}$ is designed in Figure 1. The initial state is 0, and all the states are terminal.

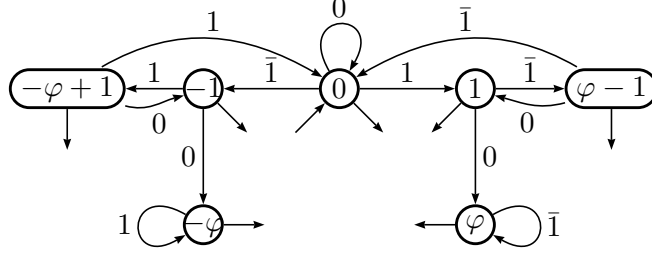


Figure 1: Finite Büchi automaton recognizing $Z_{\varphi,1}$ for $\varphi = \frac{1+\sqrt{5}}{2}$.

Notation: In the sequel $y_{m-1} \cdots y_0 \cdot y_{-1} y_{-2} \cdots$ denotes the numerical value $y_{m-1}\beta^{m-1} + \cdots + y_0 + y_{-1}\beta^{-1} + y_{-2}\beta^{-2} + \cdots$.

2.3 Numbers

A number β such that $d_\beta(1)$ is eventually periodic is a *Parry number*. It is a *simple* Parry number if $d_\beta(1)$ is finite.

A *Pisot number* is an algebraic integer greater than 1 such that all its Galois conjugates have modulus less than 1. Every Pisot number is a Parry number, see [2] and [10]. The converse is not true, see for instance Example 2 below.

3 The result

We answer a conjecture raised in [7] and obtain the following result.

Theorem 3 *The following conditions are equivalent:*

1. the set $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for every positive integer d ,
2. the set $Z_{\beta,d}$ is recognizable by a finite Büchi automaton for one $d \geq \lceil \beta \rceil - 1$,
3. β is a Pisot number.

It will be a consequence of the result which follows.

Proposition 1 *Let $d \geq \lceil \beta \rceil - 1$. If β is not a Pisot number then the set $Z_{\beta,d}$ is not recognizable by a finite Büchi automaton.*

Proof. Since β is not a Pisot number and $d \geq \lceil \beta \rceil - 1$, by Feng [4], the set

$$Y_d(\beta) = \left\{ \sum_{k=0}^n a_k \beta^k \mid n \in \mathbb{N}, a_k \in \{-d, \dots, d\} \right\}$$

is dense in \mathbb{R} . In particular there exists a sequence $(r_n)_{n \in \mathbb{N}}$ of elements of $Y_d(\beta)$ such that $r_n \neq r_m$ for all $n \neq m$, and $\lim_{n \rightarrow \infty} r_n = 0$. As r_n belongs to $Y_d(\beta)$, it can be written as $r_n = \sum_{k=0}^{\ell_n} a_k^{(n)} \beta^k$ where $a_0^{(n)}, a_1^{(n)}, \dots, a_{\ell_n}^{(n)}$ are in $\{-d, \dots, d\}$, $a_{\ell_n}^{(n)} \neq 0$ and ℓ_n is minimal.

Clearly the number

$$0.a_{\ell_n}^{(n)} a_{\ell_n-1}^{(n)} \cdots a_0^{(n)} = \frac{r_n}{\beta^{\ell_n}} \rightarrow 0 \quad (1)$$

as n tends to ∞ .

Let us fix k in \mathbb{N} . Since there exists finitely many words of length k on $\{-d, \dots, d\}$, there exists a word S_k of length k which is a prefix of infinitely many elements of the sequence $(a_{\ell_n}^{(n)} a_{\ell_n-1}^{(n)} \cdots a_0^{(n)} 0^\omega)_n$. Set $S_k = x_1 \cdots x_k$. As $a_{\ell_n}^{(n)} \neq 0$, $x_1 \neq 0$ as well.

Since infinitely many $a_{\ell_n}^{(n)} a_{\ell_n-1}^{(n)} \cdots a_0^{(n)} 0^\omega$ have S_k as a prefix, there exists a letter x_{k+1} such that $S_{k+1} = x_1 \cdots x_k x_{k+1}$ is a prefix of infinitely many $a_{\ell_n}^{(n)} a_{\ell_n-1}^{(n)} \cdots a_0^{(n)} 0^\omega$. We continue in this manner and we get an infinite string $x_1 x_2 \cdots$. To formulate an important property of it we need the following notion.

Definition 1 *Let $z_1 z_2 \cdots$ be a β -representation of 0 on $\{-d, \dots, d\}$. It is said to be rigid if $0.z_1 z_2 \cdots z_j \neq 0.0 z'_2 \cdots z'_j$ for all $j \geq 2$ and for all $z'_2 \cdots z'_j$ in $\{-d, \dots, d\}^*$.*

Claim 1 $x_1 x_2 \cdots$ is a rigid β -representation of 0.

First we prove that it is indeed a β -representation of 0: by the construction, for every $n \in \mathbb{N}$ there exists infinitely many N 's such that the common prefix of $x_1 x_2 \cdots$ and $a_{\ell_N}^{(N)} a_{\ell_N-1}^{(N)} \cdots a_0^{(N)} 0^\omega$ is longer than n . Thus

$$|0.x_1 x_2 \cdots - 0.a_{\ell_N}^{(N)} a_{\ell_N-1}^{(N)} \cdots a_0^{(N)}| \leq \frac{1}{\beta^n} \frac{2d}{\beta - 1}.$$

Therefore, by (1),

$$\begin{aligned} |0.x_1x_2\cdots| &\leq |0.x_1x_2\cdots - 0.a_{\ell_N}^{(N)}a_{\ell_N-1}^{(N)}\cdots a_0^{(N)}| + |0.a_{\ell_N}^{(N)}a_{\ell_N-1}^{(N)}\cdots a_0^{(N)}| \\ &\leq \frac{1}{\beta^n} \frac{2d}{\beta-1} + \frac{r_N}{\beta^{\ell_N}} \rightarrow 0 \end{aligned}$$

and thus $x_1x_2\cdots$ is a β -representation of 0.

Second, we prove that this representation is rigid. Suppose the opposite, that is to say that there exist some index j and a word $x'_2\cdots x'_j$ in $\{-d, \dots, d\}^*$ such that $0.x_1x_2\cdots x_j = 0.0x'_2\cdots x'_j$. There exists some N such that $a_{\ell_N}^{(N)}a_{\ell_N-1}^{(N)}\cdots a_0^{(N)}0^\omega$ has $x_1x_2\cdots x_j$ as a prefix and thus $r_N = a_{\ell_N}^{(N)}a_{\ell_N-1}^{(N)}\cdots a_0^{(N)}\cdot = 0x'_2\cdots x'_ja_{\ell_N-j}^{(N)}\cdots a_0^{(N)}\cdot$, a contradiction with the choice of ℓ_N .

Claim 2 The set $\{0.x_{n+1}x_{n+2}\cdots \mid n \in \mathbb{N}\}$ is infinite.

We show that the elements of the sequence $(0.x_{n+1}x_{n+2}\cdots)_n$ with distinct indices do not coincide. Let us suppose that there exist $k < n$ such that $0.x_{k+1}x_{k+2}\cdots = 0.x_{n+1}x_{n+2}\cdots$. Then

$$-x_1x_2\cdots x_k\cdot = -x_1x_2\cdots x_n\cdot.$$

This implies that there is some r_N such that

$$\begin{aligned} r_N &= a_{\ell_N}^{(N)}a_{\ell_N-1}^{(N)}\cdots a_0^{(N)}\cdot \\ &= x_1x_2\cdots x_na_{\ell_N}^{(N)}a_{\ell_N-n}^{(N)}\cdots a_0^{(N)}\cdot \\ &= 0^{n-k}x_1x_2\cdots x_ka_{\ell_N}^{(N)}a_{\ell_N-n}^{(N)}\cdots a_0^{(N)}\cdot \end{aligned}$$

a contradiction with the choice of ℓ_N .

Define polynomials $P_n(X) = \sum_{k=0}^{n-1} x_{n-k}X^k$. Then, the division by $(X - \beta)$ gives $P_n(X) = Q_n(X)(X - \beta) + s_n$, and

$$P_n(\beta) = s_n = x_1\beta^{n-1} + \cdots + x_{n-1}\beta + x_n = -0.x_{n+1}x_{n+2}\cdots$$

Consequently to Claim 2, the set of remainders of the division by $(X - \beta)$ is infinite. By Proposition 3.1 in [5] the set $Z_{\beta,d}$ is not recognizable by a finite Büchi automaton. \square

Remark 1 The fact that, if β is not a Pisot number, then the set $Z_{\beta,d}$ is not recognizable by a finite Büchi automaton for any $d \geq \lceil \beta \rceil$ was already settled in Theorem 1, but the proof given in Proposition 1 is more direct than the original one.

Example 2 ([7]) Let β be the root > 1 of the polynomial $X^4 - 2X^3 - 2X^2 - 2$. Then $d_\beta(1) = 2202$ and β is a simple Parry number which is not a Pisot number and has no root of modulus 1. The set $Z_{\beta,2}$ is not recognizable by a finite Büchi automaton.

Normalization in base β is the function which maps a β -representation on the canonical alphabet $A_\beta = \{0, \dots, \lceil \beta \rceil - 1\}$ of a number $x \in [0, 1]$ onto the greedy β -expansion of x . Since the set of β -expansions of the elements of $[0, 1]$ is recognizable by a finite Büchi automaton when β is a Pisot number, see[3], the following result holds true.

Corollary 1 Normalization in base β is computable by a finite Büchi transducer if and only if β is a Pisot number.

4 The case of finite representations

One can ask: what are the results in case we consider only finite representations of numbers. For instance, in the Golden Ratio numeration system, $(-1)11$ is a finite representation of 0. The situation is much simpler, since we have:

Theorem 4 ([5]) Let $\beta > 1$. The set $V_{\beta,d} = \{z_1 z_2 \cdots z_n \mid \sum_{i \geq 1}^n z_i \beta^{-i} = 0, z_i \in \{-d, \dots, d\}\}$ is recognizable by a finite automaton for every d if and only if β has no conjugate of modulus 1.

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References

- [1] D. Berend and Ch. Frougny, Computability by finite automata and Pisot bases, *Math. Systems Theory* **27** (1994) 274–282.
- [2] A. Bertrand, Développements en base de Pisot et répartition modulo 1, *C. R. Acad. Sci. Paris, Sér. A* **285** (1977) 419–421.
- [3] A. Bertrand-Mathis, Développements en base θ , répartition modulo un de la suite $(x\theta^n)_{n \geq 0}$, langages codés et θ -shift, *Bull. Soc. Math. Fr.* **114** (1986) 271–323.

- [4] D.-J. Feng, On the topology of polynomials with bounded integer coefficients, to appear in *J. Eur. Math. Soc.* arXiv:1109.1407.
- [5] Ch. Frougny, Representation of numbers and finite automata, *Math. Systems Theory* **25** (1992) 37–60.
- [6] Ch. Frougny and J. Sakarovitch, Automatic conversion from Fibonacci representation to representation in base φ , and a generalization. *Internat. J. Algebra Comput.* **9** (1999) 351–384.
- [7] Ch. Frougny and J. Sakarovitch, Number representation and finite automata, Chapter 2 in *Combinatorics, Automata and Number Theory*, V. Berthé, M. Rigo (Eds), Encyclopedia of Mathematics and its Applications 135, Cambridge University Press (2010).
- [8] A. Rényi, Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hungar.* **8** (1957) 477–493.
- [9] M. Rigo, *Formal languages, automata and numeration systems, volume 1: Introduction to combinatorics on words*, ISTE-Wiley, 2014.
- [10] K. Schmidt, On periodic expansions of Pisot numbers and Salem numbers, *Bull. London Math. Soc.* **12** (1980) 269–278.